

Day 9: Polish Kids Contest

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Problemsetters

- Franciszek Witt (Codeforces: fwitt)
- Bartłomiej Czarkowski (Codeforces: Asymmetry)
- Adam Gąsienica-Samek (Codeforces: Adam_GS)

A. Flips

Statement

You are given an $n \times m$ board of black and white cells.

Each second, any cell that belongs to any 2×2 square of diagonally-colored cells flips to the opposite.

Find the number of black cells after $t \leq 10^9$ seconds.

Once a cell starts flipping, it never stops.

Start with flipping squares and find all 2×2 squares with BFS.

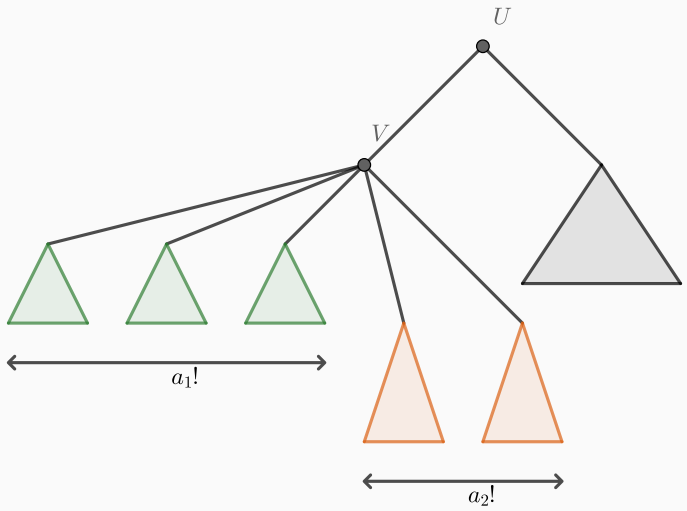
Complexity: $O(nm)$.

B. Tree permutations

Statement

Find a tree on $n \leq 400$ vertices with exactly $k \leq 10^{18}$ automorphisms.

Answer exists $\iff$ k is a product of factorials (found by backtracking).



(WLOG rooted at centroid, product over all vertices V)

Also see 102354J - Tree Automorphisms.

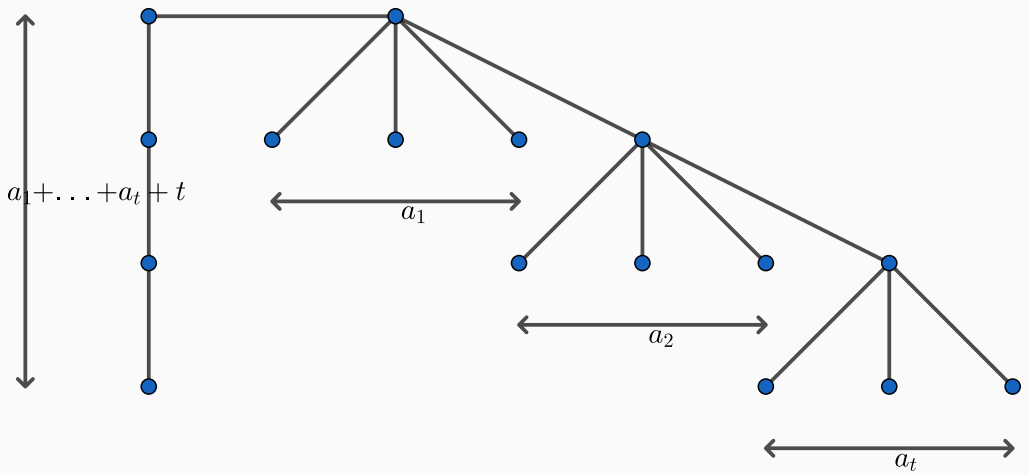
B. Tree permutations

Statement

Find a tree on $n \leq 400$ vertices with exactly $k \leq 10^{18}$ automorphisms.

Answer exists $\iff$ k is a product of factorials (factorize by backtracking).

For $k = a_1!a_2!\ldots a_t!$, at most $2(a_1 + \cdots + a_t + t)$ vertices is enough:



This needs at most $6 \log_2 n \approx 360$ vertices, for $n = 2^{59}$ in particular.

C. Control Area

Statement

There is a tree with few subtrees induced by "representative" vertices.

There are q queries:

1. The vertex v now represents subtree c;
2. Find the number of subtrees crossing the path between u and v.

Maintain each subtree as a set of its leafs in preorder.

For each vertex, maintain the number of subtrees crossing it.

To answer a query, find number of trees crossing $\mathrm{LCA}(u, v)$ and add number of subtree roots on the path from u to v.

Complexity: $O((q + n) \log(q + n))$.

D. Gray Distances

Statement

Consider the table T_n made up of Gray codes for $n \leq 2 \cdot 10^5$:

0	1	1	0	0	1	1	0
0	0	1	1	1	1	0	0
0	0	0	0	1	1	1	1

Find the total sum of distances over all pairs of cells containing 1.

The answer for T_n and the distances to the outermost 1 consist of:

1. Double the answer of T_{n-1};
2. Pairs between two T_{n-1} parts;
3. Pairs that contain 2^{n-1}.

Exact formulas are complicated, but the transition is doable in $O(n)$. Compute all in $O(n^2)$ and use intermediate values.

E. Production Line

Statement

Given an array a of length $n \leq 2 \cdot 10^5$, process $q \leq 2 \cdot 10^5$ queries:

1. Given x, y, z. If $x \leq a_i \leq y$, change a_i to z;
2. Given l, r, c. Assign $a_i = c$ for $i \in [l, r]$;
3. Given p. Find a_p;
4. Given p. Find the number of i s.t. $a_i > a_p$.

We can maintain intervals of equal values (e.g. a set of segments).

For type 1 query, also maintain DSU over intervals.

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Maintain the following:

1. Set of "interval $[l, r]$ corresponds to a value v";
2. Map from v to set of intervals corresponding to v;
3. Segtree or BIT to maintain info for type 4 queries;
4. DSU on top of v to know the color of each v.

Complexity: $O((n+q)\log^2(n+q))$ due to "small-to-large" merge.

F. Is this Fibonacci?

Statement

Given an integer of length $n \leq 10^5$, check if it's a Fibonacci number.

Pick k numbers, compare modulo them with first n Fibonacci numbers.

Complexity: $O(nk)$, probability of failure at most 2^{-k}.

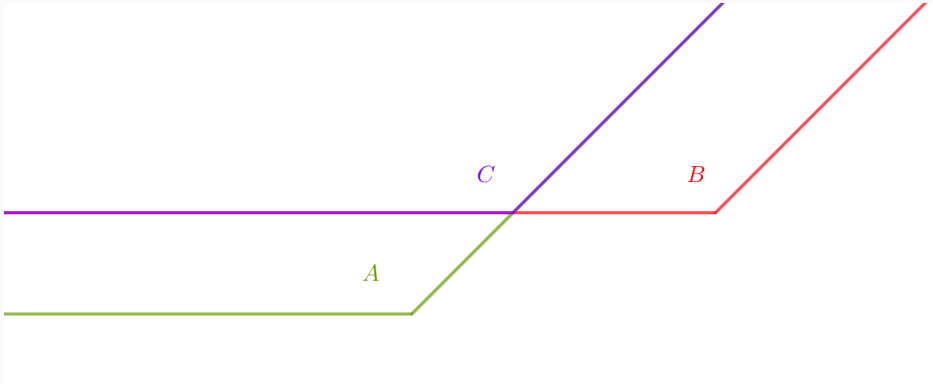
G. Queries

Statement

Process $q \leq 5 \cdot 10^5$ queries for $i \in [l, r]$:

1. Set $a_i := a_i + c$ and $b_i := \max(b_i, a_i)$;
2. Set $a_i := \max(a_i, d)$ and $b_i := \max(b_i, a_i)$;
3. Find $\max(b_l, \ldots, b_r)$.

We call $f(x) = \max(a, x + b)$ an L-function.



$\max(A, B) = C$ is an L-function

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3. Find $\max(b_l, \ldots, b_r)$.

We apply L-functions on intervals and find historic max on intervals.

We find the maximum of L-functions ever applied to the interval.

It is also an L-function, which we apply to $x = 0$ to answer the query.

Complexity: $O(q \log q)$.

H. Gray Rectangles

Statement

Consider the table T_n made up of Gray codes for $n \leq 10^{18}$:

0	1	1	0	0	1	1	0
0	0	1	1	1	1	0	0
0	0	0	0	1	1	1	1

Find the number of non-empty rectangles in T_n that contain only 1.

The answer is actually a linear recurrence sequence:

$$F_n = 9F_{n-1} - 28F_{n-2} + 36F_{n-3} - 16F_{n-4},$$
$$F_1, \ldots, F_9 = 1, 7, 32, 130, 510, 1990, 7798, 30742, 121814.$$

Can be found with Berlekamp–Massey algorithm.

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Find the number of non-empty rectangles in T_n that contain only 1.

Why is it a linear recurrence?

$$F_n = 2F_{n-1} + 2^{2(n-2)} + \sum_{i=1}^{n} \binom{2^{i-1}+1}{2}.$$

Can be maintained with matrices $\implies$ linear recurrence.

I. Coprime vertex cover

Statement

Given a graph $G = (V, E)$ with $|V|, |E| \leq 5 \cdot 10^5$.

Find a vertex cover $C \subset V$, such that $u, v \in C \implies \gcd(u, v) = 1$.

Vertex cover is a 2-SAT:

$$(u, v) \in E \implies (u \in C) \vee (v \in C).$$

For each prime p, make $\lfloor n/p \rfloor$ variables:

$$x_k = (p \in C) \vee (2p \in C) \vee \cdots \vee (kp \in C).$$

Then, $u, v \in C \implies \gcd(u, v) = 1$ is a 2-SAT:

$$(kp \in C \implies x_k), \quad (x_k \implies x_{k+1}), \quad (kp \in C \implies \neg x_{k-1}).$$

Complexity: $O(n \log n + m)$.

J. Modular Transform

Statement

You're given circular arrays $a_0, \ldots, a_{n-1} \in \mathbb{Z}_5$ and $b_0, \ldots, b_{n-1} \in \mathbb{Z}_5$.

In one operation, you can do $a_i := (a_{i-1} + a_i + a_{i+1}) \bmod 5$.

Make all elements equal in at most $8n$ operations, or report impossible.

Observations:

- There are invariant arrays, e.g. $0, 0, 0, \ldots$ or $1, 0, 4, \ldots$;
- Each operation is invertible (repeat it 4 more times);
- There is a solution $\iff$ $a = b$, or neither of them is invariant.

Solution: Find c such that a and b are easily reachable from c.

The answer is the composition $(c \to b) \circ (c \to a)^{-1}$.

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In one operation, you can do $a_i := (a_{i-1} + a_i + a_{i+1}) \bmod 5$.

Make all elements equal in at most $8n$ operations, or report impossible.

Finding c:

- Want to go over i and apply the i-th operation until $c_i = a_i$;
- On i-th step, $c'_j = a_j$ for $j < i$ and $c'_j = c_j$ for $j > i$;
- To modify c'_i, we need $c'_{i-1} + c'_{i+1} \neq 0$, thus $a_{i-1} + c_{i+1} \neq 0$;
- Need to start with i s.t. $a_{i-1} + a_{i+1} \neq 0$ and $c_{i-1} + c_{i+1} \neq 0$.

Same for b, thus any c s.t. $-c_i \notin \{a_{i-2}, b_{i-2}, c_{i-2}\}$ is fine.

Find non-circular c, then amend c_0 and c_1.

At most $4n$ operations for $c \to a$ and $c \to b$, thus at most $8n$ total.