# Reference

## 1 Definitions

These are standard, well-known definitions that will be referenced throughout the statements. They are collected here for clarity and to ensure consistency before they appear in later problems.

1. A subarray of an array $a$ is an array that can be obtained from $a$ by removing a prefix and/or a suffix of $a$. For example, if $a$ is $[2, 5, 5, 3, 1]$ then $[2]$, $[5, 3, 1]$ and $[2, 5, 5, 3, 1]$ are subarrays of $a$, while $[2, 5, 3, 1]$ is not.

2. A sequence $a$ is lexicographically smaller than a sequence $b$ if and only if one of the following holds:

   - $a$ is a prefix of $b$, but $a \neq b$; or
   - in the first position where $a$ and $b$ differ, the sequence $a$ has a smaller element than the corresponding element in $b$.

3. A permutation of length $n$ is an array consisting of the numbers $1, 2, \ldots, n$ in arbitrary order. For example, $[2, 3, 1, 5, 4]$ is a permutation, but $[1, 2, 2]$ is not a permutation (because 2 appears twice in the array), and $[1, 3, 4]$ is also not a permutation (because $n = 3$ but there is a 4 in the array).

4. A derangement $d$ is a permutation such that no element appears in its original position. In other words, if the length of the derangement $d$ is $n$, then for all $i$ such that $1 \leq i \leq n$, the condition $d_i \neq i$ holds. For example, $[2, 3, 1, 5, 4]$ is a derangement, but $[2, 1, 3]$ is not a derangement (because 3 appears in its original position in the permutation), and $[1, 3, 2]$ is also not a derangement (because $d_1 = 1$).

5. A sequence of integers $a_1, a_2, \ldots, a_n$ is strictly monotonic if either of the following holds:

   - $a_1 < a_2 < \ldots < a_n$;
   - $a_1 > a_2 > \ldots > a_n$.

6. A tree is a connected undirected graph with $n$ vertices and $n - 1$ edges.

7. A minimum spanning tree (MST) or minimum weight spanning tree is a subset of the edges of a connected, edge-weighted undirected graph that connects all the vertices together, without any cycles and with the minimum possible total edge weight. That is, it is a spanning tree whose sum of edge weights is as small as possible.

8. **Bitwise XOR**: Given two integers $x$ and $y$, consider their binary representations (possibly with leading zeros): $x_k \ldots x_2 x_1 x0$ and $y_k \ldots y_2 y_1 y_0$ (where $k$ is any number so that all bits of $x$ and $y$ can be represented). Here, $x_i$ is the $i$-th bit of the number $x$ and $y_i$ is the $i$-th bit of the number $y$. Let $r = x \oplus y$ be the result of the bitwise XOR operation of $x$ and $y$. Then $r$ is defined as $r_k \ldots r_2 r_1 r_0$ where:

$$r_i = \begin{cases} 1, & \text{if } x_i \neq y_i, \\ 0, & \text{otherwise.} \end{cases}$$

9. **Bitwise AND**: Given two integers $x$ and $y$, consider their binary representations (possibly with leading zeros): $x_k \dots x_2 x_1 x0$ and $y_k \dots y_2 y_1 y_0$ (where $k$ is any number so that all bits of $x$ and $y$ can be represented). Here, $x_i$ is the $i$-th bit of the number $x$ and $y_i$ is the $i$-th bit of the number $y$. Let $r = x \,\&\, y$ be the result of the bitwise AND operation of $x$ and $y$. Then $r$ is defined as $r_k \dots r_2 r_1 r_0$ where:

$$r_i = \begin{cases} 1, & \text{if } x_i = 1 \text{ and } y_i = 1, \\ 0, & \text{otherwise.} \end{cases}$$

10. **MEX**: The minimum excluded value of an array is the smallest non-negative integer not present in the array. For example:

    - $\text{MEX}([2, 2, 1]) = 0$ because 0 is not in the array.
    - $\text{MEX}([3, 1, 0, 1]) = 2$ because 0 and 1 are in the array but 2 is not.
    - $\text{MEX}([0, 3, 1, 2]) = 4$ because 0, 1, 2, and 3 are in the array but 4 is not.

11. A prime number is an integer $p > 1$ whose only positive divisors are 1 and $p$ itself. Equivalently, $p$ cannot be written as a product of two smaller integers greater than 1. For example, $2, 3, 5, 7, 11, \dots$ are primes.