

MITIT 2025 Spring Invitational Editorials: Qualifier Round 2

The MITIT Organizers

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1 Balls and Bins

Problem Idea: Sam Zhang

Problem Preparation: Yifan Kang

Analysis By: Albert Wang

It is apparent that as long as there are full bins, it does not matter what order in which we empty them. In particular, the only choices we make in the process is where the balls from filled bins go to.

Claim 1.1. *Let the space $e_i = s_i - a_i$ in bin i be the number of balls it can hold in addition to its current contents. Then, it is optimal to send balls to bins with less space.*

Proof. Consider an alternate version of the problem where whenever a bin is filled, we take all of its balls and keep it in reserve, from which it can be sent to any other bin arbitrarily. It is not difficult to observe that this is still equivalent to the original version of the problem.

Consider a position where we have various bins with contents a_i and capacities s_i , and r balls in reserve. If r is not at least any of the spaces $e_i = s_i - a_i$, the task becomes impossible because no additional bins may be filled. Now, let bin m be the bin with the least space.

Assume that if we moved the balls in reserve to some other bin i , the task is still possible afterwards. This means that $r \geq e_i \geq e_m$. Then, we can just fill bin e_m instead, which gives us $r + s_m$ balls in reserve; since $r + s_m \geq r \geq e_i$ still we can then empty e_i . Our new position now doesn't have bins i, m and also has $r + s_m + s_i$ balls in reserve, which is at least as solvable as the situation where we are only missing bin i and have $r + s_i$ balls in reserve. Therefore, as long as the task is possible, we can do so by prioritizing bins with as little space as possible. $\square$

With the observation in hand, it is easy to run a greedy algorithm in $O(N \log N)$ time.

2 Median of Medians

Problem Idea: Richard Qi

Problem Preparation: Benson Zhan Li Lin

Analysis By: Benson Zhan Li Lin

For subtask 1, we can brute force all possible permutations of length $3N$. For each permutation, we compute the median of medians and check if all (a_i, b_i) constraints are satisfied. This gives us an $O((3N + M) \cdot (3N!))$ time complexity.

Call a number *small* if it is strictly smaller than $\frac{3N+1}{2}$ and *large* if it is strictly greater than $\frac{3N+1}{2}$. We also say that a number k is fixed if $b_i = k$ for some i (i.e. there is a constraint that fixes k in some position). Let us break the array into the 3 segments of size N .

We will first show that a permutation is nice **if and only if** the true median (the value $\frac{3N+1}{2}$) is the median of one of the three segments.

First, note that the true median needs to be among the 3 medians of these 3 segments.

In the segment with the true median, there must be an equal number of small and large elements. This leaves us with N large elements and N small elements to be arranged into the other 2 segments. Since each of those segments has odd length, one of them will have strictly more large elements than small elements and the other will have strictly more small elements than large elements. This means that one of the segments will have a median that is large, and the other will have a median that is small. The median of these three elements must be the true median.

Thus, a permutation is nice if and only if the true median (the value $\frac{3N+1}{2}$) is the median of one of the three segments.

To gain some intuition, let's first consider the case of $M = 0$ so we can consider all possible ways to form a fitting permutation.

- There are 3 ways to pick the segment containing the median.
- There are $\binom{\frac{3N-1}{2}}{\frac{N-1}{2}}$ ways to choose the small elements that are in the same segment as the median.
- There are $\binom{\frac{3N-1}{2}}{\frac{N-1}{2}}$ ways to choose the large elements that are in the same segment as the median.
- There are $N!$ ways to arrange the elements in the segment containing the median.
- There are $(2N)!$ ways to arrange the remaining $(2N)!$ elements.

Thus the final number of fitting permutations is $3 \cdot \binom{\frac{3N-1}{2}}{\frac{N-1}{2}} \cdot \binom{\frac{3N-1}{2}}{\frac{N-1}{2}} \cdot N! \cdot (2N)!$.

To deal with the constraints, let

- s_1, s_2, s_3 be the number of fixed small numbers in the first, second and third segments respectively.
- l_1, l_2, l_3 be the number of fixed large numbers in the first, second and third segments respectively.

Then the number of free small numbers (not fixed) is $\frac{3N-1}{2} - (s_1 + s_2 + s_3)$ and the number of free large numbers is $\frac{3N-1}{2} - (l_1 + l_2 + l_3)$.

For each segment i , we want to know how many ways we can form a fitting permutation with the true median in segment i . (If the true median is fixed in some position, then we will only check that segment).

In segment i , we need to pick another $\frac{N-1}{2} - s_i$ small elements and another $\frac{N-1}{2} - l_i$ large elements. There are $\binom{\frac{3N-1}{2} - (s_1 + s_2 + s_3)}{\frac{N-1}{2} - s_i}$ ways to choose the number of small elements and $\binom{\frac{3N-1}{2} - (l_1 + l_2 + l_3)}{\frac{N-1}{2} - l_i}$ ways to choose the large elements. If either of the binomials are invalid in any way (e.g. $\frac{N-1}{2} - s_i < 0$, $\frac{3N-1}{2} - (l_1 + l_2 + l_3) < \frac{N-1}{2} - l_i$ etc.), then the number of ways is 0. The remaining elements may then be arranged arbitrarily, which is counted by another binomial expression.

Finally, we can assemble the answer by performing casework on which segment i contains $\frac{3N+1}{2}$. This gives a linear time $O(M + \max N)$ algorithm, as the answer is an expression in a constant number of binomial coefficients.

3 Busy Beaver's Colorful Walk

Problem Idea: Sam Zhang

Problem Preparation: Dylan Isaac

Analysis By: Albert Wang

It turns out that a string of length $N - 2$ may always be found.

Claim 3.1. *Consider all $N - 2$ windows of three consecutive characters in s . For each window, at least one of $\mathfrak{r}$, $\mathfrak{g}$, $\mathfrak{b}$, or $\mathfrak{y}$ is not present. Then if we take these $N - 2$ characters and concatenate them in order, the resulting string, which we will refer to as t , is a solution.*

Proof. Notice that at any point, if the x -th step of a walk w_x lies in the x -th window, our construction ensures that the walk differs from t at that position. So it suffices to show that every walk w will have some step x at which w_x lies in the x -th window. Since individual steps of the walk are not big enough to move from one side of a moving 3-character window to the other side, and the window slides from one end of the string to the other, eventually the window must be entered. $\square$

Therefore, we can just construct the string as shown in the claim and add two arbitrary characters. The overall runtime is $O(N)$.

4 Indivisible Inversions

Problem Idea: Anton Trygub

Problem Preparation: Alexander Liang

Analysis By: Albert Wang

Denote $f(i, j)$ to be the number of inversions in the subarray between indices i, j inclusive, and say that $[i, j]$ is *good* if it has a nonzero number of inversions modulo K . Consider an inversion $L < R$, $p_L > p_R$ such that $R - L$ is maximal.

Claim 4.1. *The answer is at least $R - L - 1$.*

Proof. Consider the quantity

$$f(L, R) - f(L + 1, R) - f(L, R - 1) + f(L + 1, R - 1)$$

and the contribution of each inversion towards it. Every inversion in the intervals $[L, R - 1]$ or $[L + 1, R]$ contributes zero to this sum, because it is subtracted off an equal number of times as it is added. In fact, the only inversion that has nonzero contribution is $[L, R]$, which is added once. Therefore, $f(L, R) - f(L + 1, R) - f(L, R - 1) + f(L + 1, R - 1) = 1$.

Then, considering this statement modulo K , we see that at least one of $f(L, R)$, $f(L + 1, R)$, $f(L, R - 1)$, $f(L + 1, R - 1)$ must have been nonzero modulo K , implying that the answer is at least $R - L - 1$. $\square$

Claim 4.2. *If the answer is at least $R - L$, then a longest good interval ends at 1 or N .*

Proof. Consider some good interval $[i + 1, j - 1]$ that does not contain 1 or N for $1 \leq i, j \leq N$ with length at least $R - L$. We once again consider the quantity

$$f(i, j) - f(i + 1, j) - f(i, j - 1) + f(i + 1, j - 1).$$

By similar reasoning, it is 0 if $[i, j]$ is not an inversion, and 1 otherwise. However, we know that $[i, j]$ is not an inversion, because it would otherwise contradict maximality of the inversion $[L, R]$. Therefore, $f(i, j) - f(i + 1, j) - f(i, j - 1) + f(i + 1, j - 1) = 0$. Because $[i + 1, j - 1]$ is good, it implies that $f(i + 1, j - 1) \not\equiv 0 \pmod{K}$, and therefore at least one of the intervals $[i, j]$, $[i + 1, j]$, $[i, j - 1]$ must also be good.

Therefore, any good interval $[i + 1, j - 1]$ of length at least $R - L$ may be extended until either its left endpoint is 1, or its right endpoint is N . $\square$

With the two claims in hand, we see that the desired subarray must be an interval of the forms $[L + 1, R - 1]$, $[1, x]$, $[x, N]$ for some $1 \leq x \leq N$. It is easy to check each of these cases in $O(N \log N)$ time.

5 Sequence Evaluation

Problem Idea: Jiang Cheng

Problem Preparation: Sam Zhang

Analysis By: Sam Zhang

Using induction, we can prove that for all n ,

$$a_{n+1} = \sum_{m=1}^n \sum_{s_1+\dots+s_m=n} \frac{X^m}{s_1 \cdots s_m},$$

where $s_1, \dots, s_m$ range over the positive integers. Let S be the multiset $\{s_1, \dots, s_m\}$, and let $f(S, k)$ be the number of copies of k in S . Also, let ΣS and ΠS be the sum and product of the elements of S , respectively. Then we may turn our sum into a sum over *unordered* collections S , as follows:

$$a_{n+1} = \sum_{m=1}^n \sum_{S \text{ with } |S|=m, \Sigma S=n} \frac{X^m m!}{(\Pi S) \prod_{i=1}^n f(S, i)!}.$$

But for all S with $\Sigma S = n$, $\frac{n!}{(\Pi S) \prod_{i=1}^n f(S, i)!}$ is the number of permutations on n elements with cycle type (multiset of all cycle lengths) equal to S . Taking its sum over all S with $|S| = m$ gives the number of permutations on n elements with m cycles, which is the unsigned Stirling number of the first kind¹, $\left[\begin{smallmatrix} n \\ m \end{smallmatrix} \right]$. Therefore,

$$a_{n+1} = \sum_{m=1}^n \frac{X^m m!}{n!} \left[\begin{smallmatrix} n \\ m \end{smallmatrix} \right].$$

Now let $n = P - K - 1$. It suffices to evaluate $\left[\begin{smallmatrix} n \\ 1 \end{smallmatrix} \right], \dots, \left[\begin{smallmatrix} n \\ n \end{smallmatrix} \right]$. Recall that the unsigned Stirling numbers of the first kind can be expressed as the coefficients of $x(x+1) \cdots (x+n-1)$:

$$x(x+1) \cdots (x+n-1) = \sum_{k=0}^n \left[\begin{smallmatrix} n \\ k \end{smallmatrix} \right] x^k.$$

But $x(x+1) \cdots (x+P-1) \equiv x^P - x \pmod{P}$ since $x^P - x$ is the unique degree P polynomial modulo P whose roots are $0, 1, \dots, P-1 \pmod{P}$. To obtain the coefficients of $x(x+1) \cdots (x+n-1)$, it thus suffices to start with $x^P - x$ and divide by $(x+P-K-1) \cdots (x+P-1)$, which can be done in $O(PK + P)$ time.

¹Refer to https://en.wikipedia.org/wiki/Stirling_numbers_of_the_first_kind for relevant background.